Problem Three.

There were lots of interesting solutions to this problem! I think the slickest is this. Decompose f into $f(x) = f_+(x) - f_-(x)$ in the usual way: $f_+(x) := f|_{E_+}$, where $E_+ = \{x \in \mathbb{R}^n : f(x) > 0\} \subset \mathbb{R}^n$ and $E_- = \{x \in \mathbb{R}^n : f(x) \leq 0\} \subset \mathbb{R}^n$. By the measurability of f, both E_+ and E_- are measurable.

Let's focus our attention on $f_+ : E_+ \to \mathbb{R}$. By the result of the previous pset, we know that if $\int_{E_+} f_+ = 0$, then $f_+ = 0$ a.e. We can then do the same thing for E_-. So it will suffice to show that $\int_{E_+} f_+ = 0$.

Measurable sets can be decomposed into the union of an F_σ set and a set of measure zero. So we can write $E_+ = F \cup S$, where F is F_σ and $m(S) = 0$, and

$$\int_{E_+} f_+ = \int_{E_+} f = \int_F f + \int_S f = \int_F f.$$

But F is obtained as the countable union of closed sets. Closed sets are complements of open sets. Now open sets in $\mathbb{R}^n$ can be expressed as the countable union of disjoint open boxes, together with their boundaries (which have measure zero, so we can neglect them). Therefore $\int_{open\ set} f = 0$. In particular $\int_{\mathbb{R}^n} f = 0$ as well. Thus $\int_{closed\ set} f = \int_{\mathbb{R}^n} f - \int_{open\ set} f = 0$. From this it follows that $\int_F f = 0$ (by expressing F as some suitable almost disjoint countable union of closed sets). $\square$

Problem Four.

Define $\{E_k\}_{k=1}^{\infty}$, a family of measurable subsets of $\mathbb{R}^d$, by

$$E_k := \{x \in [0,1] : \text{ the decimal expansion of } x \text{ contains a 7 at the } k\text{-th place}\}$$

and note that $m(E_k) = \frac{1}{10}$.

Let

$$E = \{x \in \mathbb{R}^d : x \in E_k, \text{for infinitely many } k\} = \text{“}\limsup\text{”}_{k\to\infty}(E_k).$$

Now, we can write

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k,$$

and therefore E is measurable.

Warning! This construction is a bit hard to think of! (As are all these weird countable unions-and-intersections things.) But it works! Think about it!

Maybe the way to say it is,

$x \in E \iff$ “**for all** $n \in \mathbb{N}$, **there exists** some $k \geq n$ such that $x \in E_k$”. That will ensure that x is in infinitely many of the E_k's.

Then the giant intersection $\bigcap_{n=1}^{\infty}$ writes down the "for all", and the giant union $\bigcup_{k\geq n}$ writes down the "there exists".

Anyway, I claim that $m(E) = 1$. To see this, consider the complement:

$$
\begin{aligned}
E^c \cap [0,1] &= \bigcup_{n=1}^{\infty} \bigcap_{k\geq n} E_k^c \cap [0,1] \\
\implies m(E^c \cap [0,1]) &= \sum_{n=1}^{\infty} m\left(\bigcap_{k\geq n} E_k^c \cap [0,1]\right) \\
&= \sum_{n=1}^{\infty} \inf_{k\geq n} \left(\frac{9}{10}\right)^{k-n} \\
&= \sum_{n=1}^{\infty} 0 \\
&= 0
\end{aligned}
$$

where we used the fact that, for this particular construction, it's always the case that E_j and E_k are "independent events" in the sense that $m(E_j \cap E_k) = m(E_j)m(E_k)$ and $m(E_j^c \cap E_k^c) = m(E_j^c)m(E_k^c)$. Note that this is not generally true, but because of the way the E_k have been chosen, it works out this time. □

Remark: Compare this problem with the second Borel-Cantelli lemma, and also the Cantor set construction.